MODELING JETS IN CROSS FLOW

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ASTRACT

Various approaches to the modeling of jets in cross flow are reviewed. These are grouped into four broad classes, namely: empirical models, integral models, perturbation models, and numerical models. Empirical models depend largely on the correlation of experimental data and are mostly useful for first-order estimates of global properties such as jet trajectory and velocity and temperature decay rates. Integral models are based on some ordinary-differential form of the conservation laws, but require substantial empirical calibration. They allow more details of the flow field to be obtained; simpler versions have to assume similarity of velocity and temperature profiles, but more sophisticated ones can actually calculate these profiles. Perturbation models require little empirical input, but the need for small parameters to ensure convergent expansions limits their application to either the near-field or the far-field. Therefore, they are mostly useful for the study of flow physics. Numerical models are based on conservation laws in partial-differential form. They require little empirical input and have the widest range of applicability. They also require the most computational resources. Although many qualitative and quantitative features of jets in cross flow have been predicted with numerical models, many issues affecting accuracy such as grid resolution and turbulence model are not completely resolved.

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Introduction

In the earliest studies of jets in cross flow, empirical models were developed to correlate experimental data obtained under various idealized conditions. Such models are reviewed in detail by Abramovich (1963), Rajaratnam (1976) and Schetz (1980). They mostly give the jet trajectory and center-line decay rates. The earliest approach based on the actual solution of conservation equations belongs to the class of integral models. These are derived either by the application of conservation principles to a finite control volume or by the use of profile assumptions to simplify the partial differential equations which describe conservation laws. These models offer more flexibility than empirical models. Jet trajectories, decay rates, growth rates, and even cross-sectional shape have been predicted. However, empirical input is usually required in the form of entrainment rates or drag coefficient. Further, it may be difficult to prescribe cross-sectional profiles in complex situations. Numerical models attempt to solve some form of the full partial differential equations, which represent conservation laws, by using a finite-difference, finite-volume or finite-element method. Little empirical input is required, hence they have the potential for the widest range of applicability. However, there may be problems with inadequate grid resolution, imprecise boundary conditions and deficiencies in the turbulence model used for closure of the mean flow equations. Recently, several models based on perturbation methods have been proposed. These are mostly of scientific interest, since drastic assumptions such as inviscid flow, negligible jet distortion, small deflection, etc., may be required for the perturbation analysis. Thus, they are used mainly to study the flow physics in limited regimes, either in the near-field or in the far-field.

In practical engineering applications, jets in cross flow are found in both confined and unconfined environments. Examples of confined jets in cross flow include: 1) Vertical and Short Take-Off and Landing (V/STOL) aircraft in transition from hover to forward flight, in which case, the jets from its engines impinge on the ground surface; 2) Internal cooling of turbine blades by air jets impinging on the leading edge, and; 3) Dilution air jets in combustion chambers of gas-turbine engines, where the jets are injected radially into the chamber, through discrete holes along its circumference, in order to stabilize the combustion process near the head, and to dilute the hot combustion products near the end.

Practical examples of jets in unconfined or semi-infinite cross flow are more numerous. These include:1) Flow situations resulting from the action of cross winds on effluents from cooling towers, chimney stacks, or flames from petrochemical plants; 2) Discharge of sewage or waste heat into rivers or oceans; 3) Film cooling of turbine blades; 4) The use of air curtains to prevent cold air from entering open spaces in industrial buildings, and; 5) Thermal plumes rising into cross winds in the atmosphere.

The configuration of a jet in cross flow is illustrated in Fig. 1. The axis of the jet is usually defined as the locus of the maximum velocity or total pressure. The jet trajectory is referred to this line, as opposed to the center-line of the jet, which is mid-way between the inner and outer boundaries of the jet, usually determined from flow visualization. The main parameter which characterizes a jet in cross flow is the jet-to-cross-flow velocity ratio, R (= U_j/U_o), or the momentum flux ratio J ($= \frac{\rho_j}{\rho_o} R^2$). In confined jets, the normalized wall distance H/D may also be important if it is not very large. In multiple jets, the normalized spacing S/D will be a factor.

As shown in Fig. 1, the jet in a cross flow has three main regions: the potential core zone (I); the zone (II) of maximum deflection; and the far-field zone (III). The potential core, in the central part of zone I remains relatively unaffected by the cross flow though its length is reduced in comparison to that of a jet in stagnant surroundings. Thus, for a turbulent jet it reduces from ($\sim$ 6D) to 6.2D $e^{-3.3/R}$ [Fan (1967), Pratte and Baines (1967)] or 6.4/(1+4.6/R) [Kamotani and Greber (1972)]. The two relations deviate at low R, where the potential core length is strongly influenced by actual exit flow conditions. In zone II, the jet experiences the most deflection. The pressure gradient across the jet is maximum as well as the entrainment rate. This is the most difficult zone to analyze accurately. In the far-field zone III, the jet axis approaches the crossflow direction asymptotically, and the flow field is nearly self-similar. All models can predict this region fairly well, given the correct boundary conditions at the end of zone II.

Four broad classes of models, namely, empirical, integral, perturbation and numerical, are now described. Emphasis will be given to the last two, where most of the recent advances have been. The first two methods were reviewed extensively in monographs by Abramovich (1963), Rajaratnam (1976) and Schetz (1980), and in a review article by Demuren (1985a).

Empirical Models

Empirical models present the simplest means of predicting global properties of jets in cross flow. They depend largely on the correlation of experimental data, and the accuracy of the predictions may depend on the closeness of the conditions of the particular problem of interest to those in the data base used for the correlation. Due to their low cost and ease of use, empirical models are most useful for first-order estimates and as qualitative checks for results produced by other methods.

The most common parameter given by empirical models is the jet trajectory. For a single circular turbulent jet injected normally into a cross flow, the trajectory has the form:

$$\frac{y}{D} = a \ J^{b} \left(\frac{x}{D}\right)^{c}$$

where, in the range of J between 2 and 2,000, a has a value between 0.7 and 1.3, b has a value between 0.36 and 0.52 and c takes a value between 0.28 and 0.40, depending on experimental conditions. The values a=0.85, b=0.47 and c=0.36 appear to be a good compromise for the intermediate range of J. This equation should also be valid for confined jets, up to the point of contact, and for multiple jets with medium to large spacing ratios. Equation (1) with b = 0.36, and c = 0.28 also gives the physical boundaries of jets in cross flow [Pratte and Baines (1967)]; with a = 1.35 and 2.63 for the inner and outer boundaries, respectively, and a = 2.05 for the center line. For plane jets in confined cross flow Kamotani and Greber (1974) found that Eq. (1) can be used with $a = 2.0\ (1–e^{–H/D})$, b = 0.28 and c = 0.50. Equations for other parameters such as entrainment rates, velocity profiles, temperature trajectories, etc., can be found in Demuren (1985a).

Integral Models

Integral models are the first elaborate calculation procedures applied to predict the behavior of jets in cross flow. In these models, integral equations are derived either by considering the balance of forces and momentum changes over an elementary control volume of the jet, or by integrating in two spatial directions, the three-dimensional, partial differential equations governing the jet flow. In either case, a set of ordinary differential equations is obtained which can be solved analytically or numerically. Empirical input is required to prescribe pressure drag, entrainment rates and spread rates. The former approach is easier to understand and to implement and is therefore more popular. On the other hand, the latter approach involves more extensive mathematical manipulation, but it is more transparent in the assumptions made and affords more flexibility in dealing with complex boundary conditions and trajectories.

Integral models flourished between the late 1960's, when more flexibility was required than could be obtained with empirical models and the early 1980's, when the rapid growth in computer hardware and software made elaborate numerical computations of three-dimensional flows feasible. Many of the earlier model are reviewed by Rajaratnam (1976). In these models, there was an assumption of the constancy of the momentum in either the initial jet direction, the cross flow direction or the axial direction, and the jet was bent over by a prescribed pressure drag force, or entrainment of ambient fluid. None of these models could predict correctly the jet trajectory over a range of R [Demuren (1985a)]. Thus, they offer no advantage over much simpler empirical models.

More refined integral models consider effects of both the pressure drag and the entrainment of cross flow ambient fluid on the jet. A typical model is that proposed by Fan (1967) for buoyant jets in cross flow. The cross section of the jet was assumed circular with radius $\sqrt{2}\ \mathrm{b}$, and the excess velocity profile was assumed to be Gaussian, i.e., $\mathrm{V} - \mathrm{U_o}\cos\theta = (\mathrm{V_{max}} - \mathrm{U_o}\cos\theta)\mathrm{e}^{-\eta^2/\mathrm{b}^2}$. The resulting set of ordinary differential equations can be written as:

Continuity

$$\frac{d}{d\zeta}\left(\rho A\bar{V}\right) = C\rho_o U_e$$

x-momentum

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\rho \mathrm{A}\lambda_{\mathrm{v}}\bar{\mathrm{V}}^2\cos\theta\right) = \mathrm{C}\ \rho_{\mathrm{o}}\mathrm{U_e}\mathrm{U_o} + 0.5\ \mathrm{C_D}\triangle \mathrm{z}\ \rho_{\mathrm{o}}\mathrm{U_o^2}\sin^3\theta$$

y-momentum

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\rho \mathrm{A}\lambda_{\mathrm{v}}\bar{\mathrm{V}}^2\sin\theta\right) = -\mathrm{A}(\rho - \rho_{\mathrm{o}})\mathrm{g} + 0.5\ \mathrm{C_D}\triangle \mathrm{z}\ \rho_{\mathrm{o}}\mathrm{U_o^2}\sin^2\theta\cos\theta$$

Scalar (Temperature/Concentration) ϕ

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\rho\ \mathrm{A}\ \lambda_\phi\bar{\mathrm{V}}\ \phi\right) = 0$$

where A is the cross-sectional area of the jet, C its circumference, ρ is the density of the jet fluid, g the acceleration due to gravity, and λ_v and λ_ϕ are respectively, momentum or scalar flux coefficients which depend on the assumed velocity and scalar profiles. U_e is the entrainment velocity. Fan proposed that it should be proportional to the velocity vector difference between the jet and the cross flow, but this was found to be unreliable. Abraham (1971) proposed an entrainment model with two parts as:

$$\mathrm{U_e} = \mathrm{E_{mom}}(\mathrm{V_{max}} - \mathrm{U_o}\cos\theta) + \mathrm{E_{th}}\mathrm{U_o}\sin\theta\cos\theta$$

where the coefficients E_{mon} and E_{th} have the values 0.057 and 0.50, respectively. The first part of Eq. (6) represented the entrainment of a momentum jet in a nearly stagnant ambient fluid and the second part the entrainment into (momentum-free) thermals under similar conditions. Cosθ was introduced artificially into the second part to prevent it from contributing to entrainment when the jet was nearly perpendicular to the cross flow. With this entrainment model, the drag coefficient C_D was given a value of 0.3. Equations (2) through (6) were then integrated numerically. With this model, Abraham (1971) was able to obtain quite good agreement with experimental data of jet trajectory and axial concentration decay. Typical results are shown in Fig. 2. For a non-buoyant jet, E_{th} = 0 and $\rho = \rho_o$ so that the buoyancy term in Eq. (4) is also zero.

Equations (2) through (6) may also be applied to predict plane jets in cross flow by substituting the appropriate expressions for the area A and the circumference C. The entrainment coefficients E_{mom} and E_{th} and the drag coefficient C_D must then be calibrated with plane jet data.

In order to extend the range of applicability of integral models to include more flow physics and be able to deal with more practical situations, such as multiple jets in varied arrangements, highly non-uniform cross flow, etc., more elaborate models have been proposed by Campbell and Schetz (1973), Isaac and Schetz (1982), Makihata and Miyai (1983), amongst others. All these models were derived based on the control volume approach and are applicable mainly to jets with plane trajectories. Hirst (1972) and Schatzmann (1979) developed models based on the integration in the cross plane of the jet of the three-dimensional partial differential equations by making the assumption of axi-symmetry and that profiles of the excess velocity are Gaussian. These latter models could be applied to situations with three-dimensional jet trajectories.

Although integral models allow economical prediction of several flow properties in comparison to full-blown numerical models, they have been criticized for the need to assume the shape of the jet cross-section and profile functions, some of which may not be realistic for the whole evolution of the jet in cross flow, especially in the zone of maximum deflection. However, it appears that in spite of the apparent oversimplification, integral models can be made to perform well in some cases with proper calibration. Adler and Baron (1979) proposed a quasi-three-dimensional model which did not assume the cross-sectional shape or similarity profiles for the velocity, but these were computed along with other flow variables. The characteristic kidney-shaped cross-section of the jet was computed successfully by considering the evolution of vortices distributed along the boundaries of the jet in a Lagrangian manner, and the cross-section was allowed to grow at a rate which was an average between the growth rates of free jets and vortex pairs. Similarly, velocity profiles were allowed to change in the zone of maximum deflection, culminating in self-similar profiles only in the far-field zone (III). The model gives quite good prediction of the three-dimensional flow fields of jets in cross flow studied experimentally by Kamotani and Greber (1972).

Perturbation Models

If in the jet in cross flow problem a small parameter is defined, perturbation methods can be used to solve the governing equations. Most applications of perturbation models have been to study the flow of strong jets in a weak cross flow. In the initial stage, the flow can be considered to be a small perturbation from that of a free jet in stagnant surroundings, and the jet stiffness λ (= 1/R) can be used as the small parameter. This places a severe restriction on the range of applicability of such models. However, they have the advantage of not being too dependent on empirical calibration as integral methods are, and they are computationally much cheaper than numerical methods. The goal is to predict the main features of jets at high R (>10), including trajectory, cross-sectional shape, velocity field, vorticity field, mixing, etc., with minimal empirical input. It was believed [Needham et al. (1988), (1990)] that the jet distortion and deflection could be obtained by inviscid analyses based on the evolution of vortex filaments around the jet as it exits from a circular pipe or orifice. This approach was based on the original work of Chen (1942) which was also the basis of fairly successful computations with the integral model of Adler and Baron (1979).

Chen's model approximates the near-field as two regions of irrotational flow, the jet flow and the external cross flow, separated by a vortex sheet. The three-dimensional vortex sheet can then be approximated by a two-dimensional vortex sheet, which originates from the pipe or orifice exit and evolves in time in the axial direction. Needham et al. (1988), (1990) applied a three-dimensional model, with perturbation expansions for the potential flow within and outside of the jet. The distortion of the jet could be predicted reasonably well, but contrary to earlier studies, no jet deflection was obtained, if the jet issued normally into the cross flow. Surprisingly, with a component of the cross flow in the direction of the jet, some deflection was obtained. This discrepancy was explained by Coelho and Hunt (1989) who showed that the two-dimensional time-evolving vortex sheet model was a poor approximation for the fully three-dimensional vortex sheet model.

The two-dimensional vortex sheet equation can be written as

$$\frac{\partial \gamma}{\partial t} + \frac{\partial}{\partial s}(U_s \gamma) = 0$$

where γ is the vortex strength and U_s is the average speed of the flow across the layer, using the nomenclature of Fig. 3(a). If γ (θ, t), U_s (θ, t) and $R(\theta, t)$ are approximated as Taylor series expansions with respect to t, derivatives of γ, U_s and R of any order with respect to t, at t=0, can be evaluated. This gives the shape of the vortex sheet or the jet boundary as

$$R(\theta, t) = R_o - \left[\frac{1}{2}\frac{U_o^2}{R_o}\cos 2\theta\right] t^2 + \left[\frac{1}{6}\frac{U_o^3}{R_o^2}\{3\cos 3\theta - \cos\theta\}\right] t^3 + \mathrm{O}\left(t^4\right)$$

If it is assumed that elements of the vortex-sheet travel at half-speed, such that $t = \frac{2y}{U_j}$, and U_j and R_o are used for normalization, then Eq. (8) becomes:

$$R(\theta, y) = 1 - \left[2y^2 \cos 2\theta\right]\lambda^2 - \left[\frac{4}{3}y^3\{3\cos 3\theta - \cos\theta\}\right]\lambda^3 + O\left(\lambda^4\right)$$

However, in the presence of the cross flow, the flow in the jet pipe is distorted [Andreopoulos (1983)], so that U_j is not uniform, and Eq. (9) may not be a good approximation for Eq. (8). The actual non-uniformity of the jet exit flow can be calculated using a fully three-dimensional vortex-sheet model. In this case, the longitudinal and transverse components of vorticity must be considered. As shown in Fig. 3(b), these can be approximated by the vertical and azimuthal components γ_y and γ_s, respectively. The general expression for the strength of the vortex sheet can be written in vector form as

$$\frac{\partial}{\partial t}\vec{\gamma} + \left(\vec{\mathbf{U}}_{\mathbf{v}} \cdot \vec{\nabla}\right)\vec{\gamma} = \left(\vec{\gamma} \cdot \vec{\nabla}\right)\vec{\mathbf{U}}_{\mathbf{v}} - \vec{\gamma}\left(\vec{\nabla} \cdot \vec{\mathbf{U}}_{\mathbf{r}}\right)$$

where $\vec{\gamma} = \{\gamma_s, \gamma_y\}, \vec{\mathbf{U}}_{\mathbf{v}} = \{U_s, U_y\}$ and $\vec{\nabla} = \left\{\frac{\partial}{\partial s}, \frac{\partial}{\partial y}\right\}$. The vertical component of Eq. (10) is

$$U_y\frac{\partial \gamma_y}{\partial y} + \frac{\partial}{\partial s}(U_s\gamma_y) = \frac{\partial U_y}{\partial s}\gamma_s$$

which contains a source term, in contrast to Eq. (7). This source term expresses the rate at which fluid elements rotate as they travel up the vortex sheet. Thus, the vertical vortex strength may be strongly influenced by the azimuthal vortex strength and variations in the azimuthal velocity. The solution of the three-dimensional vortex-sheet problem, in terms of the potential flow inside the jet, and the external potential flow gives the shape of the vortex-sheet to third order as

$$R(\theta, y) = 1 - \lambda^2\left[y^2 - 2C_2y - \sum_{n=1}^{\infty} A_nJ_3(\sigma_n)e^{-\sigma_n y} - 1\right]\cos 2\theta + \mathrm{O}\left(\lambda^4\right)$$

where C_2 is a constant and A_n are coefficients given by the boundary conditions. $J_3(\ )$ are Bessel functions of third order, and σ_n are zeros of $J_2\ (\ )$. Solutions for the velocity field (in terms of the velocity potential) and the pressure field are also given in terms of Bessel functions. Comparison of Eqs. (9) and (12) shows that the O (λ^3) term in the former, which produces the deviation from symmetry and thus the jet deflection is absent in the latter. Therefore, Coelho and Hunt (1989) concluded that a three-dimensional inviscid vortex-sheet model could not produce jet deflection. Although, the two-dimensional model of Chen (1942) could produce a deflection, this was an incorrect approximation of the three-dimensional flow. However, by introducing a vertical component to the cross flow, the symmetry in Eq. (12) may be broken and a second small parameter is introduced into the perturbation expansion, as in the works of Needham et al. (1988), (1990). Then, jet deflection would occur.

Coelho and Hunt (1989) postulated that viscous or turbulent entrainment was necessary for jet deflection. They proposed an entraining vortex-sheet model [see Fig. 3 (c) for nomenclature]. The entrainment velocity from the external cross flow is given by

$$V_{ce} = e\left(\gamma_s^2 + \gamma_y^2\right)^{1/2} + \mathrm{O}\ \left(e^2\right)$$

where e is the entrainment coefficient, which must be prescribed empirically, and it now becomes a second small parameter for the perturbation expansion. The mixing layer within the vortex-sheet entrains fluid from both the jet and the external cross flow. To the leading order, the entrainment rates are assumed proportional, so that

$$V_{je} = e\ c\left(\gamma_s^2 + \gamma_y^2\right)^{1/2} + \mathrm{O}\left(e^2\right)$$

where c is a constant of O (1). The equation for the conservation of mass within the mixing layer is

$$\frac{\partial}{\partial\theta}[U_\theta(R_e - R_j)] + \frac{\partial}{\partial y}\left[U_y\left(\frac{R_e^2 - R_j^2}{2}\right)\right] = e\left(\gamma_s^2 + \gamma_y^2\right)^{1/2}$$
$$\left\{\left[R_e^2 + \left(\frac{\partial R_e}{\partial\theta}\right)^2\right]^{1/2} + c\left[R_j^2 + \left(\frac{\partial R_j}{\partial\theta}\right)^2\right]^{1/2}\right\} + \mathrm{O}\left(e^2\right)$$

Equation (15) must be solved along with the potential flow equations for the jet flow and the external flow. The solution yields for the mean radius

$$R(\theta, y) = 1 + ye + \left[(1+c)y^2 cos\theta\right]e\lambda - [Z(y)cos2\theta]\lambda^2$$
$$+\mathrm{O}\left(e^2, \lambda^3, e\lambda^2\right)$$

where $Z(y) = y^2 - 2C_2 y - \sum\limits_{n=1}^{\infty} A_n J_3(\sigma_n)\left[e^{-\sigma_n y} - 1\right]$. Comparison of eqs. (12) and (16) shows that the deviation from symmetry is now of O(eλ), so that jet deflection would occur, as one would normally expect.

Higuera and Martinez (1993) have proposed a mixed perturbation/numerical model which does not use the vortex-sheet concept but solves the parabolized Navier-Stokes (PNS) equations in the distortion region of jets in weak cross flow (R>15). The model is applicable to laminar flow or a turbulent flow in which the assumption of a constant eddy viscosity would be appropriate. The weak cross flow is necessary so that there is only mild curvature in the distortion zone II, enabling the governing equations to be parabolized. Furthermore, there should be little deviation of the jet flow in the development zone I from that of a free jet, so that the flow field at the end of zone I and the beginning of zone II can be prescribed from Landau's self-similar solution for a point source of momentum [Batchelor (1967)]. A perturbation method is used to solve the PNS equations for small y, with y as the small parameter. For intermediate values of y, a parabolic numerical method is used. However, computations must be stopped once the distortion becomes too large for the assumptions of negligible axial diffusion and pressure gradient to be valid.

For small y = [O(4y_w)], where y_w = [($\sqrt{3}$/16)RD], the deflection of the jet, β, will be small. The continuity and momentum equations, in dimensionless variables, can then be written for the axial velocity component, *v* and cross stream velocity vector $\vec{V}$ = {*u*,*w*} as

$$\vec{\nabla} \cdot \vec{\mathrm{V}} + \frac{\partial v}{\partial y} = 0$$

$$v\frac{\partial v}{\partial y} + \left(\vec{V} \cdot \vec{\nabla}\right)v = \vec{\nabla}^2 v + \beta^2\left(-\frac{\partial P}{\partial y} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$v\frac{\partial \vec{\mathbf{V}}}{\partial y} + \left(\vec{\mathbf{V}} \cdot \vec{\nabla}\right)\vec{\mathbf{V}} = -\vec{\nabla}P + \vec{\nabla}^2\vec{\mathbf{V}} + \beta^2\frac{\partial^2 \vec{\mathbf{V}}}{\partial y^2}$$

where $\vec{\nabla} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\}$ and $\vec{\nabla}^2 = \left\{\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial z^2}\right\}$. The requirement that y_w should be beyond the development zone I indicates how large R must be for the analysis to be valid. For example for turbulent flow, with a development length ≈6D, R > 15. Therefore $\beta \to 0$, and terms in β^2 can be neglected. If the pressure gradient term is also eliminated by combining the divergence of Eq. (19), with the continuity equation, the governing equations become

$$v\frac{\partial v}{\partial y} + \left(\vec{\mathbf{V}} \cdot \vec{\nabla}\right)v = \vec{\nabla}^2 v$$

$$v\frac{\partial \Omega}{\partial y} + \left(\vec{\mathbf{V}} \cdot \vec{\nabla}\right)\Omega = \Omega\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial v}{\partial x} + \nabla^2\Omega$$

$$\Omega = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$$

where Ω is the vertical component of vorticity. Equations (20) – (22) are parabolic in y, so they can be solved by marching in zone II.

The initial conditions are derived from Landau's self-similar profiles as

$$\left(yu, yw, yv, y^2\Omega\right) \rightarrow \left(V_{rs}\cos\theta,\, V_{rs}\sin\theta,\, v_s,\, 0\right) \text{ as } y \rightarrow 0$$

where,

$$V_{rs} = \frac{4\eta\left(1-\eta^2\right)}{\left(1+\eta^2\right)^2}\ , v_s = \frac{8}{\left(1+\eta^2\right)^2}\ , \eta = \frac{r}{y}$$

The boundary conditions as x $\rightarrow \infty$ are

$$v = w = \Omega = 0\ ;\ u \rightarrow 1$$

the near-field solution has the form

$$\left(v, v_r, v_\theta, y\Omega\right) = \frac{1}{y}\left(v^{-1}, v_r^{(-1)}, 0, 0\right) + \left(v^{(0)}, v_r^{(0)}, v_\theta^{(0)}, \Omega^{(0)}\right)$$

$$+y\left(v^{(1)}, V_r^{(1)}, V_\theta^{(1)}, \Omega^{(1)}\right) + \ldots\ldots$$

The functions on the right hand side of Eq. (25) depend only on η and θ, whereas Eqs. (20) – (22) are functions of η, θ and y. Hence, by substituting Eq. (25) into Eqs. (20) – (22) and collecting terms of like order in y, solutions of different order can be found. Of course, terms of order (-1) will reproduce the initial conditions. This approach is really quite restrictive. The several requirements of large R, small y, constant turbulent eddy viscosity and low Reynolds number exclude it from consideration as a realistic tool for practical computations of jets in cross flow.

In general, perturbation models are not yet sufficiently mature to become more than curious tools of analysis. A redeeming factor is that it is especially in those high R flows, for which they are valid, that most numerical models are least accurate. In these flows, there are substantial regions with high shear and rates of strain, in which standard discretization schemes and turbulence models may become inadequate.

Numerical Models

Numerical models have the most potential for wide generality and can, in principle, be applied to the whole range of jet in cross flow situations, confined or unconfined, low medium or high R, single or multiple jets, impinging on a wall or on other jets, swirling, homogeneous or heterogeneous cross flow, compressible or incompressible, etc. The analysis starts from the general conservation laws stated in partial differential equation form, which are the Navier-Stokes equations for the velocity field, and corresponding energy or species equations for the temperature or concentration fields, respectively. These equations, which describe unsteady, three-dimensional flow cannot be solved directly in practical applications for turbulent flows. In incompressible fluid flow, time-averaged forms, and in compressible fluid flow, density-weighted, time-averaged (or Favre-averaged) forms of the equations are solved. The process of time-averaging introduces a closure problem due to non-linear correlation between fluctuating velocity and/or temperature/concentration fields. Turbulence models are required to determine these correlations, thereby affecting closure of the system of equations. Most numerical models applied to the jet in cross flow problem use the eddy viscosity concept. In its simplest form, the turbulent eddy viscosity is prescribed as a constant, whereas more sophisticated models solve partial differential equations for turbulent quantities, from which the eddy viscosity distribution can then be obtained. Experimental studies by Andreopoulos and Rodi (1984) show that there are significant regions of the jet cross flow interactions in which the eddy viscosity concept is invalid. Demuren (1992) proposed a numerical model in which the eddy viscosity concept is not invoked but partial differential equations are solved to determine distributions of the turbulent correlations directly. In most numerical models, the computational domain encompasses the whole region in which the influence of the jet is felt, or if necessary the whole field of the jet and cross flow. No assumptions are required as to the evolution of the jet within the flow domain, but this is obtained as a result of the computations. It is only necessary to prescribe boundary conditions at the chosen computational boundaries. The two major issues in the application of numerical models to jets in cross flow are the accuracy of the basic numerical method and the accuracy of the turbulence model.

The time-averaged, three-dimensional, steady-state mean flow equations can be written in Cartesian tensor notation as

continuity

$$\frac{\partial}{\partial x_l}(\rho U_l) = 0$$

momentum

$$\frac{\partial}{\partial x_l}(\rho\, U_i U_l) = -\frac{\partial}{\partial x_i}P + \frac{\partial}{\partial x_l}\left[-\rho\, \overline{u_i u_l} + \mu\left(\frac{\partial U_i}{\partial x_l} + \frac{\partial U_l}{\partial x_i}\right)\right]$$

scalar

$$\frac{\partial}{\partial x_l}(\rho\, U_l \phi) = S_\phi + \frac{\partial}{\partial x_l}\left[-\rho\, \overline{u_l \vartheta} + \frac{\mu}{\sigma}\frac{\partial \phi}{\partial x_l}\right]$$

with i = 1, 2, 3 and l = 1, 2, 3 representing properties in the lateral, vertical and longitudinal directions, respectively. The equations are expanded with Einstein's summation rule for repeated indices. x_i are the Cartesian coordinates and U_i the Cartesian velocity components. ϕ may represent any scalar such as the temperature or species concentration. $-\rho\overline{u_i u_l}$ and $-\rho\overline{u_l \vartheta}$ represent the Reynolds stresses and the turbulent scalar fluxes, respectively. Distributions of these quantities are obtained from the turbulence model. Also, μ is the molecular viscosity and σ the corresponding Prandtl of Schmidt number. S_ϕ is the source term for the temperature or concentration equation.

The task of the turbulence model is to provide distributions for the Reynolds stresses and the scalar fluxes so that the mean flow Eqs. (26) to (28) can be closed. In the Boussinesq eddy viscosity concept, the Reynolds stresses are calculated from

$$-\rho\overline{u_i u_l} = \mu_t\left(\frac{\partial U_i}{\partial x_l} + \frac{\partial U_l}{\partial x_i}\right) - 2/3\ \rho k \delta_{il}$$

The corresponding eddy diffusivity concept gives

$$-\rho\, \overline{u_l \vartheta} = \frac{\mu_t}{\sigma_\phi}\frac{\partial \phi}{\partial x_l}$$

where μ_t is the turbulent eddy viscosity, σ_ϕ is the turbulent Prandtl or Schmidt number, k is the turbulent kinetic energy (per unit mass) and δ_{il} is the Kronecker delta which is equal to unity when $i = l$, and zero otherwise.

The most common method for calculating the distribution of μ_t is through the k-ϵ turbulence model [Launder and Spalding (1974)]. This gives

$$\mu_t = c_\mu \; \rho \frac{k^2}{\epsilon}$$

The distributions of k and ϵ are then obtained from solution of transport equations which can be written in Cartesian tensor form as

$$\frac{\partial}{\partial x_l}(\rho \; U_l k) = \frac{\partial}{\partial x_i}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_l}\right) + \rho \; P_k - \rho\epsilon$$

$$\frac{\partial}{\partial x_l}(\rho \; U_l \epsilon) = \frac{\partial}{\partial x_l}\left(\frac{\mu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_l}\right) + c_{\epsilon_1}\rho \; P_k \frac{\epsilon}{k} - c_{\epsilon_2}\rho\frac{\epsilon^2}{k}$$

where ϵ is the rate of dissipation of k, and P_k is the rate of production of k through the interaction of the Reynolds stresses with the mean flow. It is given by

$$P_k = -\overline{u_m u_l}\frac{\partial U_l}{\partial x_m}$$

The empirical coefficients which appear in Eqs. (29) — (33) are given the standard values c_μ = 0.09, c_{ϵ_1} = 1.44, c_{ϵ_2} = 1.92, σ_ϕ = 0.9, σ_k = 1.0 and σ_ϵ = 1.3.

Simpler eddy viscosity relations have been utilized with reasonable success in some studies. Chien and Schetz (1975) prescribed a constant value for μ_t, proportional to the jet velocity excess and the jet diameter. However, in a subsequent study Oh and Schetz (1990), calculated μ_t from a relation which takes into consideration the complex shape of the jet and the relative magnitude of the axial turbulence intensity to the velocity excess. Thus

$$\mu_t = 0.037 \; f \; \rho \; b_{1/2} \; \triangle U_c$$

where $b_{1/2}$ is the characteristic half width of the jet, ΔU_c the centerline velocity excess, and f (= $\overline{u'^2}/\triangle U_c^2$) takes a value of 2 in the potential core and $\{1 + \exp[-1.134(\zeta - \zeta_o)]\}$ in the main jet region. For computations at high R, Sykes et al. (1986) proposed a one-equation model which solves the k equation such as Eq. (32), but calculates the Reynolds stresses from

$$-\rho \; \overline{u_i u_l} = \rho \; k^{1/2}\Lambda\left(\frac{\partial U_i}{\partial x_l} + \frac{\partial U_l}{\partial x_i}\right)$$

where Λ is a length scale given by

$$\Lambda = 0.088\mathrm{D} + 0.0088\mathrm{r}$$

D is the jet diameter and r is the distance from the center. In spite of the rather crude length scale assumption, computed results, of the mean flow (for R=2) agreed reasonably with experimental data. However, turbulent kinetic energy levels were grossly overpredicted, especially in the wake region.

Demuren (1992), (1994) and Alvarez and Jones (1993) have used various Reynolds stress models (RSM) to investigate the effect of the turbulence model on computations of jets in cross flow. Model computations in which the eddy viscosity concept is not invoked but partial differential equations are solved for the Reynolds stresses are compared to those using the k-ϵ model and to experimental data. The Reynolds stress equations can be written in Cartesian tensor notation as

$$\frac{\partial}{\partial x_l}(U_l\, \overline{u_i u_j}) = D_{ij} + P_{ij} + \pi_{ij} - \epsilon_{ij}$$

where D_{ij} is the turbulent diffusion, P_{ij} is the production, π_{ij} is the pressure-strain correlation and ϵ_{ij} the dissipation rate. The production term $P_{ij} = -\overline{u_i u_l}\, \frac{\partial U_j}{\partial x_l} - \overline{u_j u_l}\, \frac{\partial U_i}{\partial x_l}$, and the dissipation rate is assumed to be locally isotropic so that $\epsilon_{ij} = 2/3\ \delta_{ij}\epsilon$. D_{ij} and π_{ij} contain higher-order correlations, and so must be approximated for closure at this level. In Demuren (1992), these terms are modeled after proposals of Daly and Harlow (1970) (denoted by DH) and Launder, Reece and Rodi (1975) (denoted by LRR), respectively. In Demuren (1994) and Alvarez and Jones (1993), additional models for D_{ij} and π_{ij} are considered, including those proposed by Mellor and Herring (1973) (denoted by MH) and Speziale, Sarkar and Gatski (1991) (denoted by SSG), respectively. The latter combination of models was found to give the best overall predictions of developed turbulent plane channel flow [Demuren and Sarkar (1993)]. The DH and MH diffusion models can be written, respectively, as

$$D_{ij} = Cs_1 \frac{\partial}{\partial x_k}\left(\frac{k}{\epsilon}\, \overline{u_k u_l}\, \frac{\partial \overline{u_i u_j}}{\partial x_l}\right)$$

$$D_{ij} = Cs_2 \frac{\partial}{\partial x_k}\left[\frac{k^2}{\epsilon}\ \left(\frac{\partial \overline{u_i u_j}}{\partial x_k} + \frac{\partial \overline{u_i u_k}}{\partial x_j} + \frac{\partial \overline{u_j u_k}}{\partial x_i}\right)\right]$$

with C_{s1} = 0.22 and C_{s2} = 0.072. The pressure-strain models can be written in the general form

$$\pi_{ij} = \alpha_o \, \epsilon \, b_{ij} + \alpha_1 \, \epsilon \left(b_{ik} b_{kj} - \frac{1}{3} II \delta_{ij} \right) + \alpha_2 \, k \, S_{ij}$$
$$+ \alpha_3 \, P_k \, b_{ij} + k\{ \alpha_4 \left(b_{ik} S_{jk} + b_{jk} S_{ik} - \frac{2}{3} \delta_{ij} b_{kl} \, S_{kl} \right)$$
$$+ \alpha_5 \left(b_{ik} W_{jk} + b_{jk} W_{ik} \right) \}$$

where $b_{ij} \equiv \left(\frac{\overline{u_i u_j}}{2k} - \frac{1}{3} \delta_{ij} \right)$ is the Reynolds stress anisotropy tensor, $S_{ij} \equiv \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$ is the rate of strain tensor, $W_{ij} \equiv \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right)$ is the rotational tensor, and $II = b_{lk} b_{kl}$ is the second invariant of b_{ij}. For the LRR model, $\alpha_o = -3 + f_w$, $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = 0.8$, $\alpha_4 = 1.745$ and $\alpha_5 = 1.309 - 0.24 \, f_w$. For the SSG model, $\alpha_o = -3.4$, $\alpha_1 = 4.2$, $\alpha_2 = 0.8 - 1.3 \, II^{1/2}$, $\alpha_3 = -1.8$, $\alpha_4 = 1.25$ and $\alpha_5 = 0.40$. f_w is a wall proximity function which has a value of unity near a wall and zero in a turbulent flow free from walls. The correct rate of decay away from walls is a subject of controversy [Demuren and Rodi (1984)]. It is also difficult to specify in complex flows with curved walls or multiple walls. The absence of such a term makes the SSG pressure-strain model rather attractive for application to complex flows.

A turbulence modeling approach which is intermediate between the k-ϵ model and the full Reynolds stress model was utilized by Baker et al. (1987) to calculate the near field of jets in cross flow at high R. This is the so-called algebraic Reynolds stress model, which is derived by dropping the convection and diffusion terms in Eq. (38). Thus, implicit algebraic equations are obtained which can be solved simultaneously for the Reynolds stresses. Baker et al. (1987) used further simplifications of these equations to obtain explicit expressions for the Reynolds stresses. These expressions contain k and ϵ as unknowns so that Eqs. (32) and (33) must still be solved before the Reynolds stresses can be calculated. This approach falls within the general class of non-linear k-ϵ models reviewed by Speziale (1991).

The boundary conditions applied to the equations depend on the particular problem. The various types of boundaries which may exist in these flow situations are inflow, outflow, wall, symmetry planes and the free stream. At inflow boundaries, the values of the dependent variables are prescribed or deduced from experimental data. At outflow planes, boundary conditions such as zero traction force or zero normal gradient are usual prescribed. It is normal to prescribe no-slip conditions along walls, but in order to bridge the flow between the fully turbulent region and the viscous sublayer near the wall, the wall-function method [Launder and Spalding (1974)], which is based on the assumption of local equilibrium, is usually employed to prescribe variable values along the first set of grid nodes nearest to the wall. Kim and Benson (1992) did not use this approach. Rather a two-layer model was used to integrate the equations all the way down to the wall. Sykes et al. (1986) and Oh and Schetz (1990) avoided this problem entirely by using slip conditions at the wall. Along symmetry planes, zero normal velocity and zero normal gradients for other variables are usually prescribed. Additionally, Reynolds shear stresses with a component in the normal direction will also be zero. Along the free stream, known variable values or zero surface stresses are prescribed. A major uncertainty exists as to the proper boundary condition at the jet exit plane. Experimental data by Andreopoulus (1983) had indicated that exit conditions are highly modified by the cross flow, especially at low R. Demuren (1983) found that by specifying constant total pressure at the jet exit, axial velocity profiles, similar to those observed experimentally could be simulated. However, in-plane velocity profiles had to be prescribed empirically. Kim and Benson (1992) overcame the uncertainty by placing the inflow boundary one diameter into the pipe from which the jet flow originates, and fully developed pipe flow conditions were prescribed at the recessed boundary. A different treatment would be required if the jet exited through an orifice rather than a pipe.

Equations (26) – (41) form closed sets which can be solved by a finite-difference, finite volume or finite-element method to yield the mean flow and turbulence fields. By far the most popular approach is a combination of finite difference and finite volume methods. These are different manifestations and extensions of numerical techniques originally proposed by Chorin (1968), and Patankar and Spalding (1972). Notable exceptions are the finite-element computations of Baker et al. (1987) and Oh and Schetz (1990).

Current numerical models can, in principle, be used to predict most flows of jets in cross flow which occur in practice. Both laminar and turbulent flows can be computed, so long as the flow is not dominated by rapid distortion or coherent structures. Instabilities develop in jet flows for Reynolds numbers greater than O (10), so that in most practical situations the flow will be turbulent or transitional. The former can be handled by current models with proper choice of the turbulence model. In the latter, large scale coherent structures may play a dominant role

[Andreopoulos (1985)], and methods based on Reynolds averaging may be inadequate. Direct calculation of the unsteady flow will generally be required, but this cannot be done for any practical Reynolds number, so a large eddy simulation appears to be a viable option. Such approaches are reviewed in Galperin and Orszag (1993), but much research work is still required before they can become reliable predictive tools in applications of interest here. In addition, current turbulence models, cannot predict higher-order statistics such as the Reynolds stresses [Speziale (1994)] in flows with rapid distortion (ratio of turbulent-to-mean-flow time scales greater than 50), or in flows with high compressibility (free shear flows with turbulent Mach number of order 1). Of course, the question of the importance of such higher-order statistics in the present flows has been raised. Some studies indicate that they are less important at high R, where pressure effects dominate, than at low to medium R.

Earlier model computations of the fully elliptic type such as by Patankar et al. (1977) used grids (15x15x10 in the x, y, z directions) which are too coarse for the results to be considered reliable. Although correct trajectories were predicted, velocity fields deviated qualitatively and quantitatively from experimental observations. Finer grid computations have been reported, but grid independence could not be demonstrated conclusively in any of these. Multigrid methods [Demuren (1992)] allow systematic studies of grid dependence since on each finer grid level there are twice as many points in each direction than on the coarser one, i.e. eight times as many total grid nodes in three dimensions. Although up to 2.4 million nodes (i.e., 256x96x96 on the 5th grid-level) were used, Claus and Vanka (1992) could still not demonstrate grid independence of the computed velocity and turbulence fields. Figure 4 shows comparisons of vertical profiles of the streamwise velocity component computed on the three finest grids. It is obvious that grid-convergence has not been achieved, especially in the near-field. Similar comparisons are shown for the turbulence level in Fig. 5. Far-field results appear closer to grid-convergence. With this type of grid refinement, better estimate of the results can be obtained by using Richardson extrapolation techniques [Demuren and Wilson (1994)]. The question of grid resolution cannot be completely separated from that of the formal order of accuracy of the numerical scheme. Most studies of the Patankar and Spalding (1972) type have used the hybrid upwind/central difference scheme to approximate convection terms. This is monotonic and conservative, but it is known to be highly diffusive. [Demuren (1985b)]. Studies in which higher-order differences, such as the quadratic-upstream-weighted (QUICK) scheme were utilized e.g. Barata et al. (1991)] showed that similar results, as with lower-order schemes could be obtained on coarser grids. However, higher-order schemes tend to suffer from lack of boundedness in regions with high gradients.

The uncertainty in the specification of boundary conditions for the jet hole exit has been discussed. Figure 6 shows contours of the jet velocity, static pressure and total pressure at the exit plane computed by Kim and Benson (1992). None of these is uniform, which is a compelling reason for including the jet pipe hole in the calculation domain. Other uncertainties involve the specification of inflow and near-wall conditions for turbulent quantities.

In many computational studies, the inadequacy of the turbulence model has been blamed for the lack of agreement between computed results and experimental data. Demuren (1992) tried to isolate the effect of the turbulence model by performing computations on the same grid with the k-ϵ model and the Reynolds stress (LRR-DH) model. The results are compared to experimental data of Atkinson et al. (1982) for opposed jets in cross flow in Figs. 7 and 8, for the mean flow and Reynolds stress fields, respectively. For the mean flow, there is little to choose between both model predictions, but the Reynolds stress model clearly gives better predictions of Reynolds stress profiles. From these results, it may be concluded that the mean flow was not strongly influenced by the turbulence field. Thus, if the interest is solely in the mean flow field, the k-ϵ turbulence model, or an even simpler model, would be adequate. But, if the turbulence field is required, e.g., to predict mixing, then a Reynolds stress model would give much better results, but at additional computational cost. The multigrid technique enables the additional cost to be minimized by ensuring nearly grid-independent convergence rate. Thus, for a 3–level multigrid scheme, with (42x34x82) points on the finest grid, convergence of the Reynolds stress model computations could be obtained in less than 100 fine grid iterations.

Concluding Remarks

In the study of turbulent jets in cross flow, empirical models offer quick and simple methods for obtaining first-order estimates and a qualitative picture of the jet trajectory, its extent, and decay rates of its axial velocity and temperature. The main requirement for reasonable predictions is the use of correlation equations or curves derived from experimental data bases with similar characteristics as the problem of interest.

Integral models contain in simplified forms mathematical representations of the basic conservation laws, and can thus be applied much more widely than empirical models. Several physical phenomena which occur in the flow are modeled with relations which are more or less empirical. Combinations of these have been used successfully in integral models, so long as they are properly calibrated. One criticism of integral models is that they provide little insight into flow physics, since the same effect could be achieved in several different ways. All integral models are computationally cheap to use. The basic models are conceptually simple, but more sophisticated models have been devised which enable more complex jet cross flow interactions to be analyzed.

Perturbation models do not require much empirical input, but they are mostly restricted to the near-field or far-field where small parameters required for expansions can be defined. They enable order-of-magnitude studies of the effects of various parameters, and are thus useful tools for the investigation of flow physics. Beyond these, they have limited practical utility.

Numerical models offer the best choice as practical predictive tools over a wide range of jets in cross flow applications. They require the least assumptions and empirical input. They are, however, the most computationally intensive. Quite complex jet-jet, jet-cross flow interactions can be analyzed. Depending on specific requirements, the choice of turbulence model may or may not be important. Although, several quite complex flow situations have been computed with a measure of quantitative accuracy, some questions remain as to the effects of grid resolution, turbulence model and boundary conditions on overall accuracy of computed results. Reliability and computational accuracy are expected to improve with further developments in numerical techniques and turbulence models. These are clearly the models of choice for the computation of practical jets in cross flow situations. Computer codes are available commercially for this purpose.

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FIGURE CAPTIONS

Figure 1 : Configuration of a jet in cross flow

Figure 2 : Prediction of jet trajectories and concentration decay; symbols — experimental data (from Fan, 1967), curves — calculations (from Abraham, 1971).

Figure 3 : Nomenclature for perturbation model description (from Coelho and Hunt, 1989); (a) 2D vortex sheet model, (b) 3D vortex sheet model, (c) entraining 3D vortex sheet model.

Figure 4 : Grid dependency test: comparison between computations from Claus and Vanka (1992) and experimental data from Khan et al. (1982) of streamwise velocity in the center plane; (a) x/D = 4, (b) x/D = 8.

Figure 5 : Grid dependency test: comparison between computations from Claus and Vanka (1992) and experimental data from Crabb et al . (1981) of turbulence intensity in the center plane; (a) x/D = 2, (b) y/D = 1.35.

Figure 6 : Contours plots of the flow field at the jet exit plane (from Kim and Benson, 1992); (a) axial velocity ($U_j/\bar{U}_j$), (b) static pressure ($P/\frac{1}{2}\rho\bar{U}_j^2$), (c) total pressure $[(P+\frac{1}{2}\rho U_j^2)/\frac{1}{2}\rho\bar{U}_j^2]$.

Figure 7 : Effect of turbulence model: comparison between computations from Demuren (1992) and experimental data from Atkinson et al. (1982) of streamwise velocity in the center plane; (a) flow configuration – opposed jets in cross flow, (b) x/D = 8, (c) x/D = 12.

Figure 8 : Effect of turbulence model: comparison between computations from Demuren (1992) and experimental data from Atkinson et al. (1982) of Reynolds stresses in the center plane; (a) x/D = 8, (b) x/D = 12.